Backpropagation

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Gradient Descent

Network parameters $\theta = \{w_1, w_2, \cdots, b_1, b_2, \cdots\}$

Starting Parameters $\theta^0 \longrightarrow \theta^1 \longrightarrow \theta^2 \longrightarrow \ldots\ldots$

$$\nabla L(\theta) = \begin{bmatrix} \partial L(\theta)/\partial w_1 \\ \partial L(\theta)/\partial w_2 \\ \vdots \\ \partial L(\theta)/\partial b_1 \\ \partial L(\theta)/\partial b_2 \\ \vdots \end{bmatrix}$$

$Compute\ \nabla L(\theta^0)$ $\quad \theta^1 = \theta^0 - \eta \nabla L(\theta^0)$

$Compute\ \nabla L(\theta^1)$ $\quad \theta^2 = \theta^1 - \eta \nabla L(\theta^1)$

Millions of parameters ……

To compute the gradients efficiently, we use ***backpropagation***.

Chain Rule

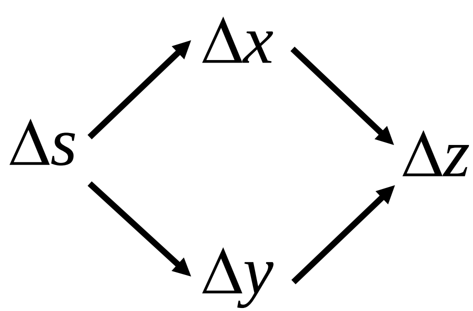
Case 1

$$y = g(x) \qquad z = h(y)$$

$$\Delta x \rightarrow \Delta y \rightarrow \Delta z \qquad\qquad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

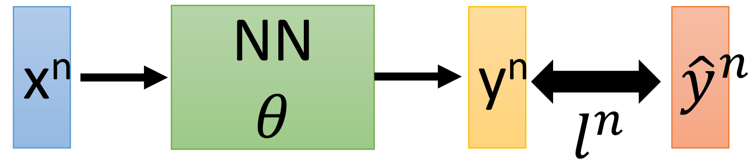
Case 2

$$x = g(s) \qquad y = h(s) \qquad z = k(x, y)$$

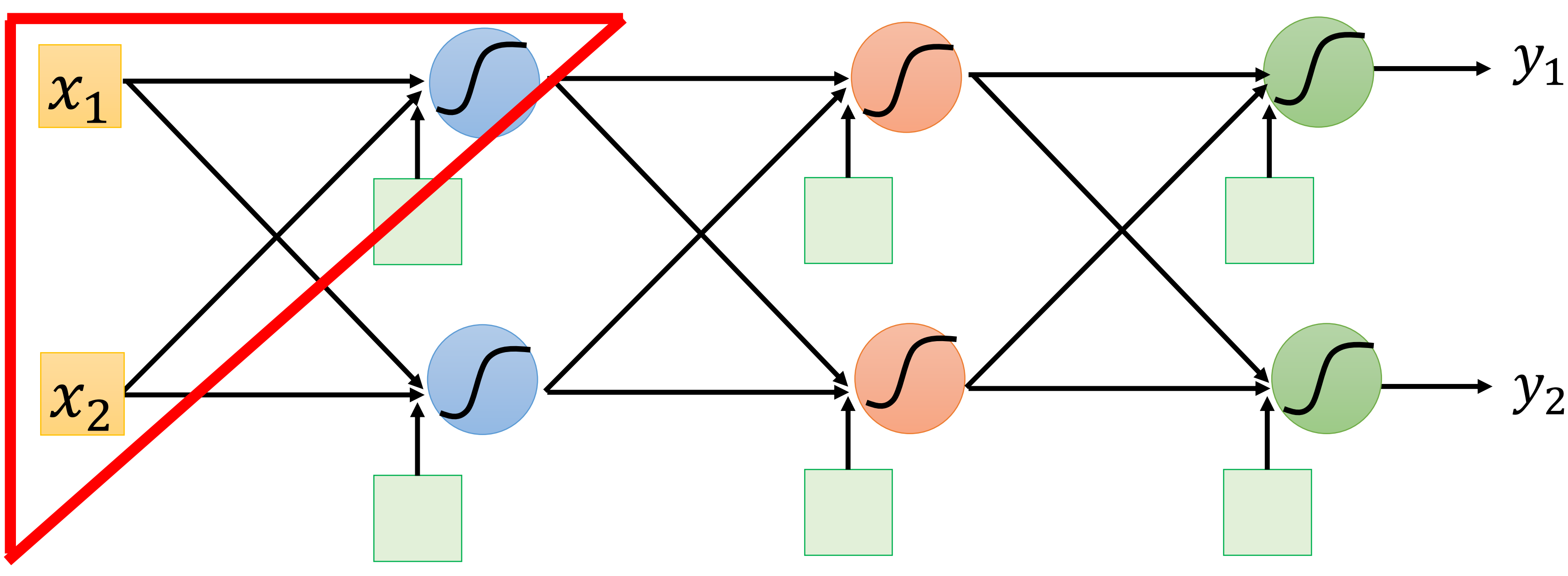


$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

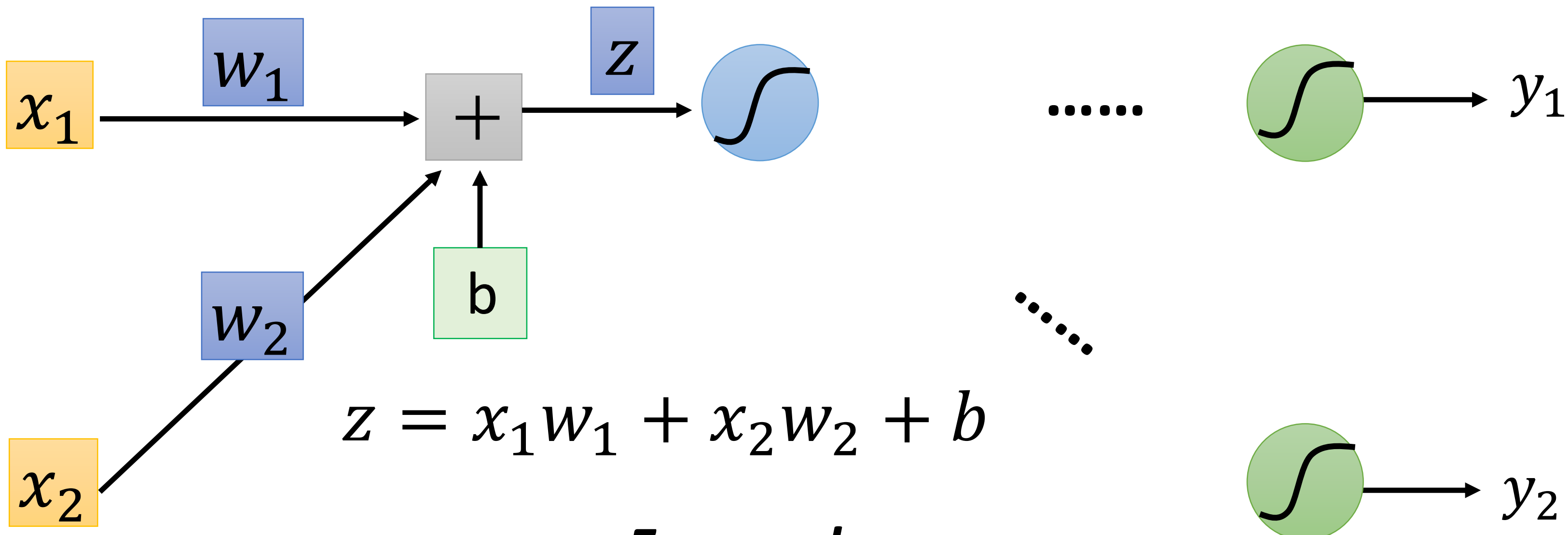
Backpropagation



$$L(\theta) = \sum_{n=1}^{N} l^n(\theta) \quad \Rightarrow \quad \frac{\partial L(\theta)}{\partial w} = \sum_{n=1}^{N} \frac{\partial l^n(\theta)}{\partial w}$$



Backpropagation



$$z = x_1 w_1 + x_2 w_2 + b$$

$$\frac{\partial l}{\partial w} = ? \quad \frac{\partial z}{\partial w} \frac{\partial l}{\partial z}$$

(Chain rule)

Forward pass:

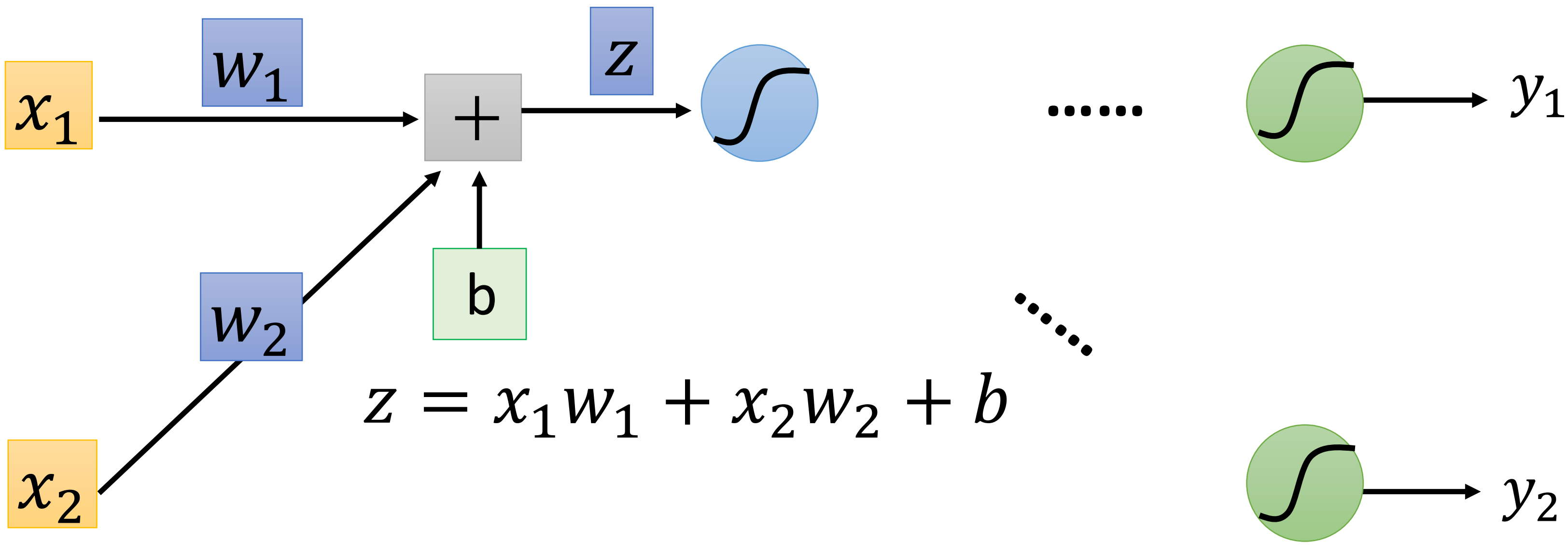
Compute $\partial z / \partial w$ for all parameters

Backward pass:

Compute $\partial l / \partial z$ for all activation function inputs z

Backpropagation – Forward pass

Compute $\partial z / \partial w$ for all parameters



$$z = x_1 w_1 + x_2 w_2 + b$$

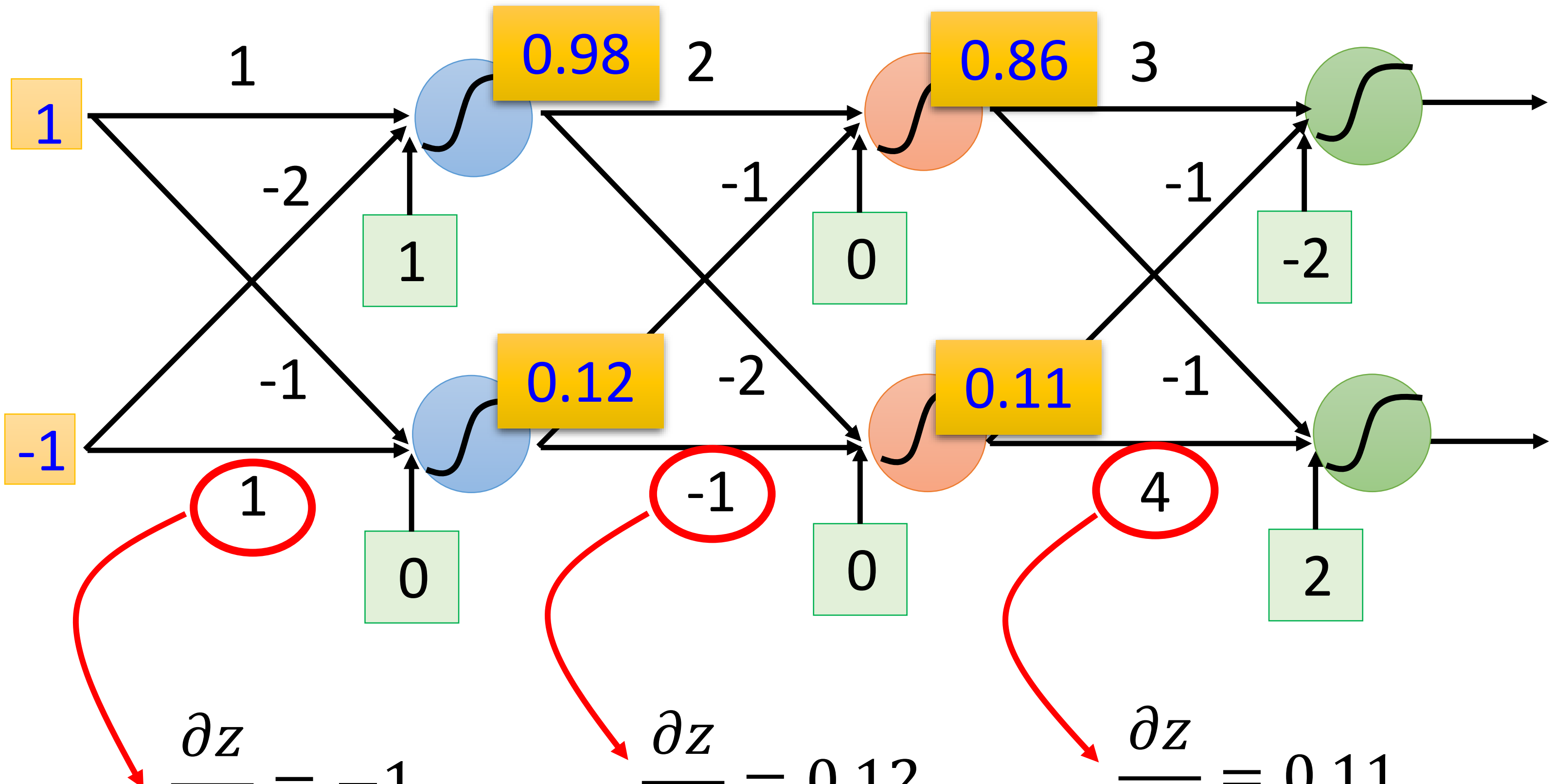
$\partial z / \partial w_1 = ?\ \ x_1$

$\partial z / \partial w_2 = ?\ \ x_2$

The value of the input connected by the weight

Backpropagation – Forward pass

Compute $\partial z / \partial w$ for all parameters



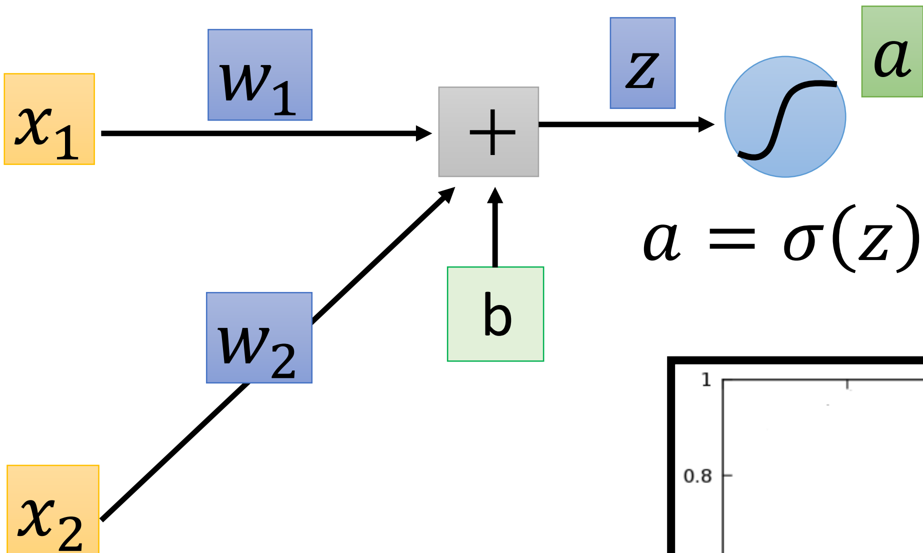
$$\frac{\partial z}{\partial w} = -1$$

$$\frac{\partial z}{\partial w} = 0.12$$

$$\frac{\partial z}{\partial w} = 0.11$$

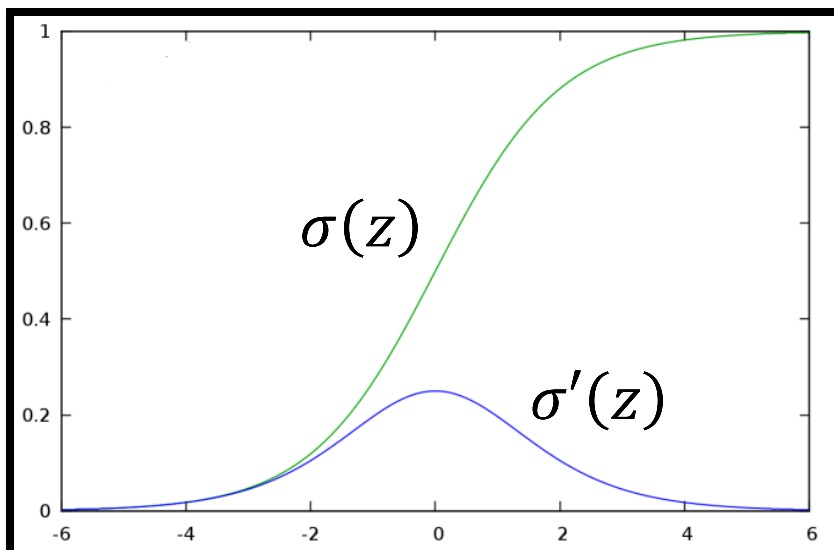
Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z



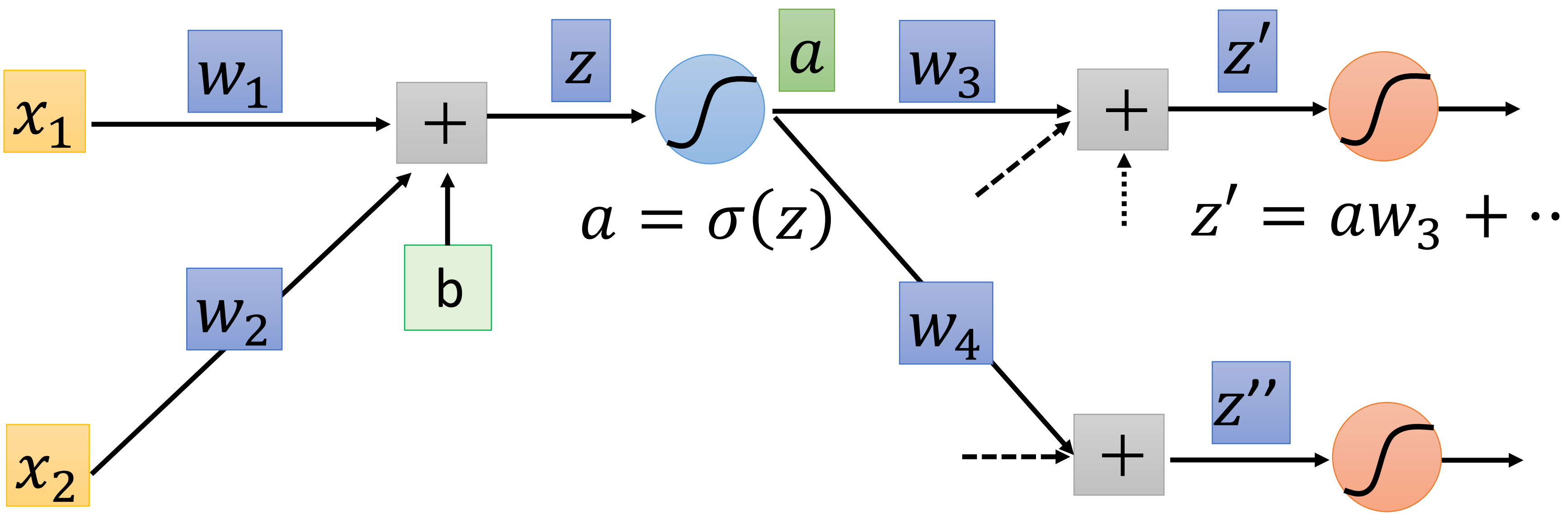
$$\frac{\partial l}{\partial z} = \frac{\partial a}{\partial z}\frac{\partial l}{\partial a}$$

→ $\sigma'(z)$



Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z



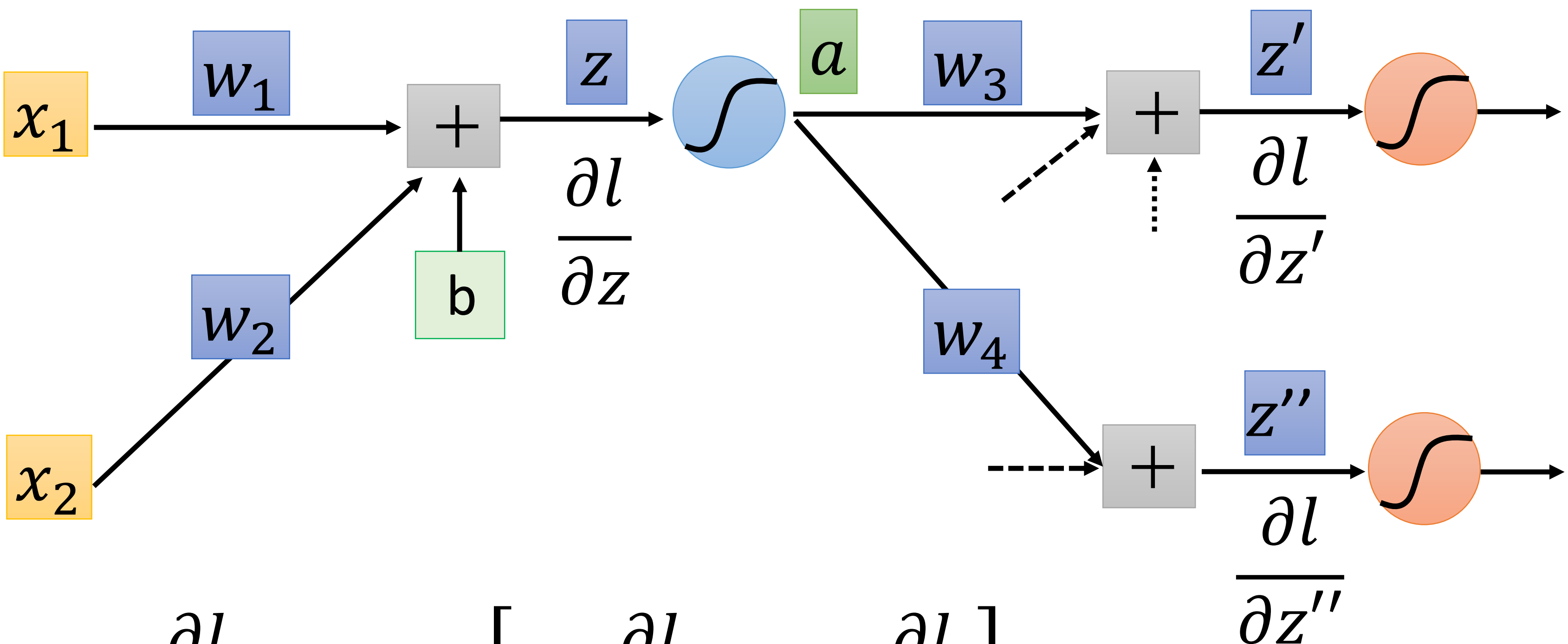
$$\frac{\partial l}{\partial z} = \frac{\partial a}{\partial z}\frac{\partial l}{\partial a} \qquad \frac{\partial l}{\partial a} = \frac{\partial z'}{\partial a}\frac{\partial l}{\partial z'} + \frac{\partial z''}{\partial a}\frac{\partial l}{\partial z''} \quad \text{(Chain rule)}$$

w_3 ? w_4 ?

Assumed it's known

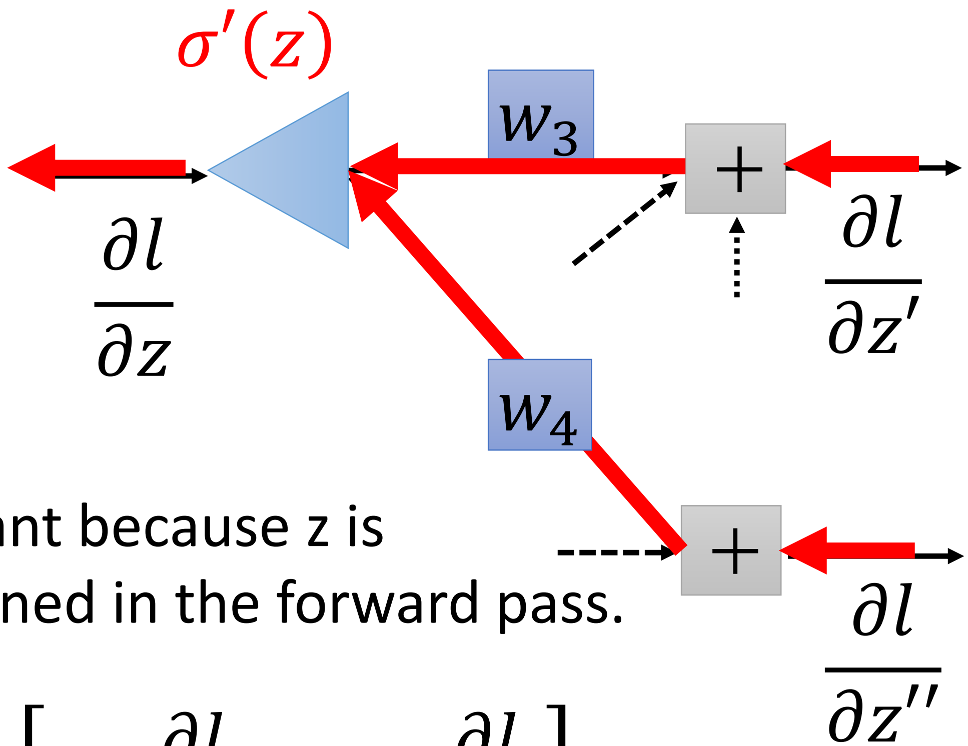
Backpropagation – Backward pass

Compute $\partial l/\partial z$ for all activation function inputs z



$$\frac{\partial l}{\partial z} = \sigma'(z)\left[w_3 \frac{\partial l}{\partial z'} + w_4 \frac{\partial l}{\partial z''}\right]$$

Backpropagation – Backward pass

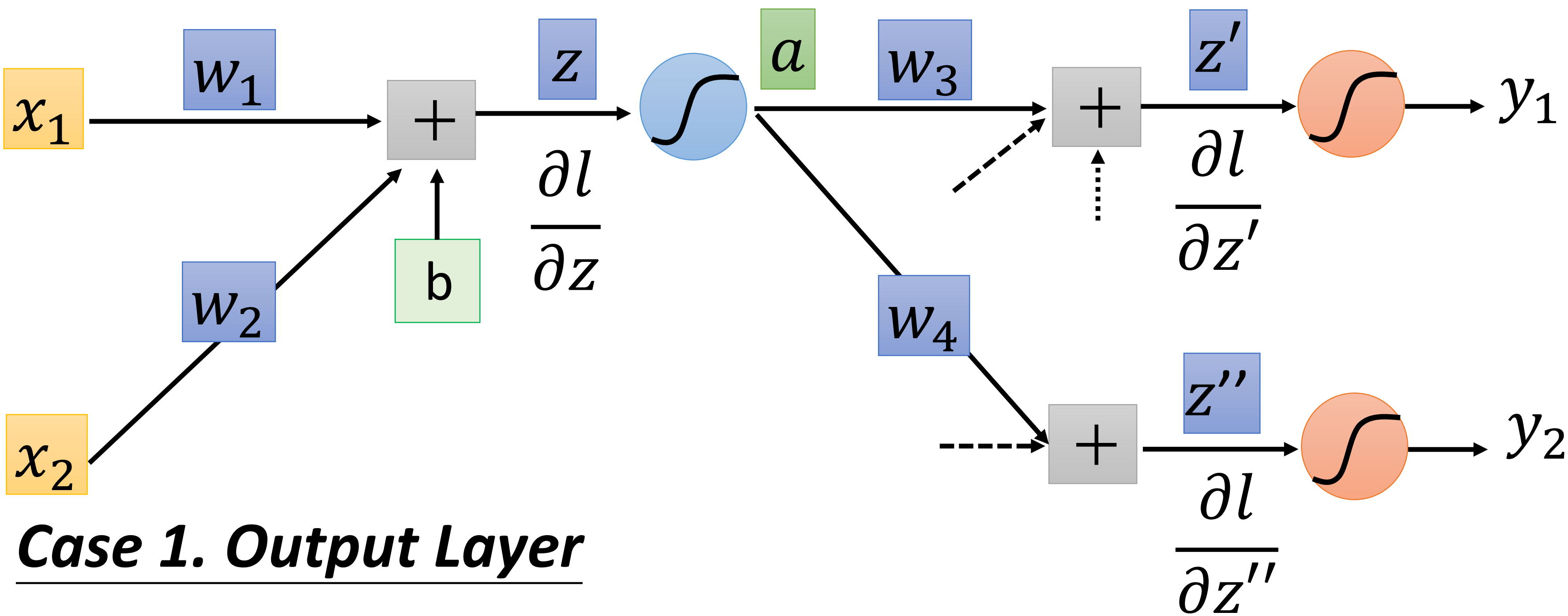


$\sigma'(z)$ is a constant because z is already determined in the forward pass.

$$\frac{\partial l}{\partial z} = \sigma'(z)\left[w_3 \frac{\partial l}{\partial z'} + w_4 \frac{\partial l}{\partial z''}\right]$$

Backpropagation – Backward pass

Compute $\partial l/\partial z$ for all activation function inputs z



Case 1. Output Layer

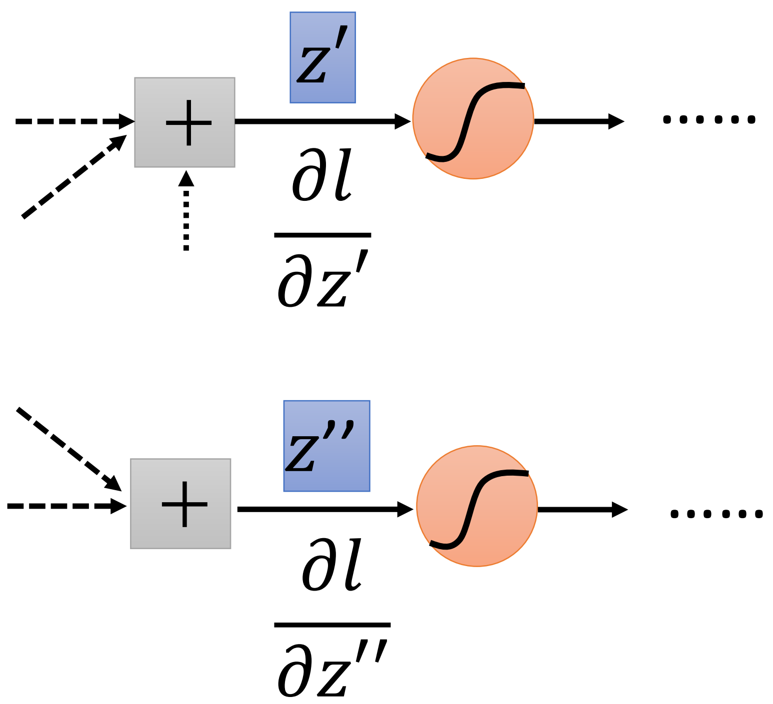
$$\frac{\partial l}{\partial z'} = \frac{\partial y_1}{\partial z'}\frac{\partial l}{\partial y_1} \qquad \frac{\partial l}{\partial z''} = \frac{\partial y_2}{\partial z''}\frac{\partial l}{\partial y_2}$$

Done!

Backpropagation – Backward pass

Compute $\partial l/\partial z$ for all activation function inputs z

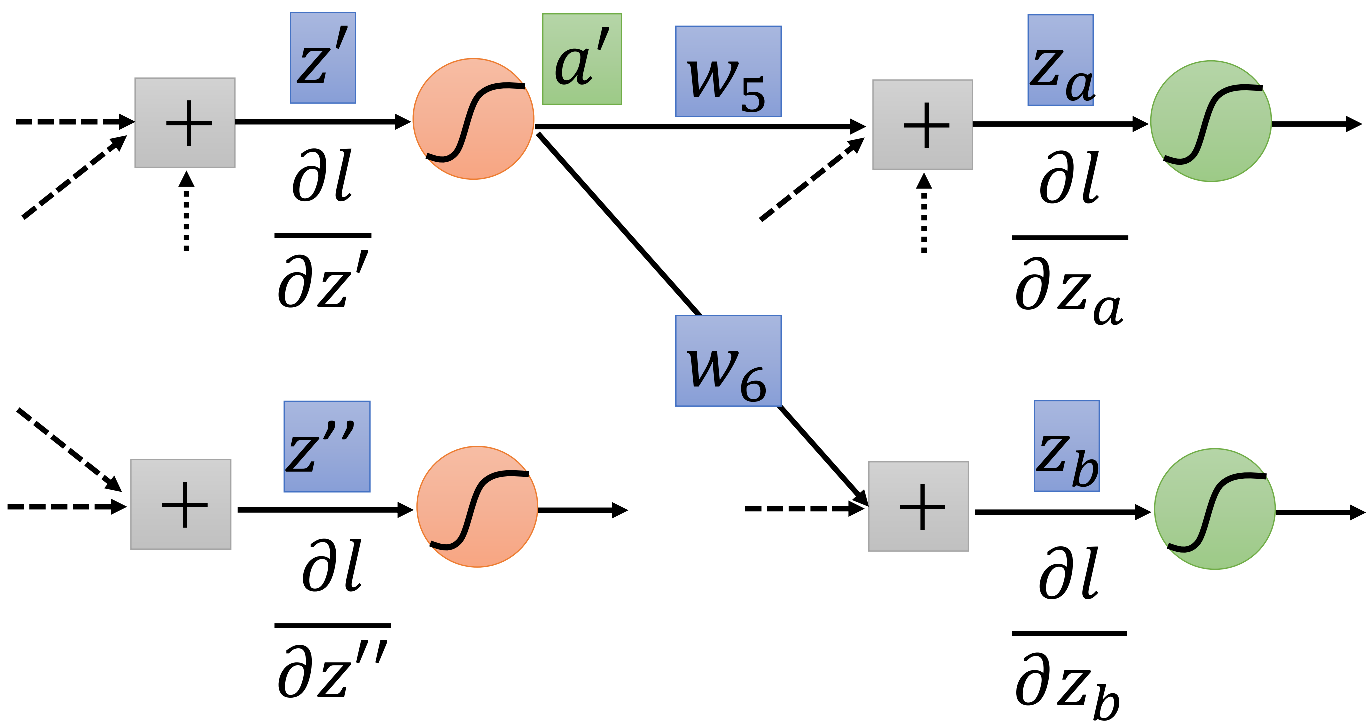
Case 2. Not Output Layer



Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z

Case 2. Not Output Layer



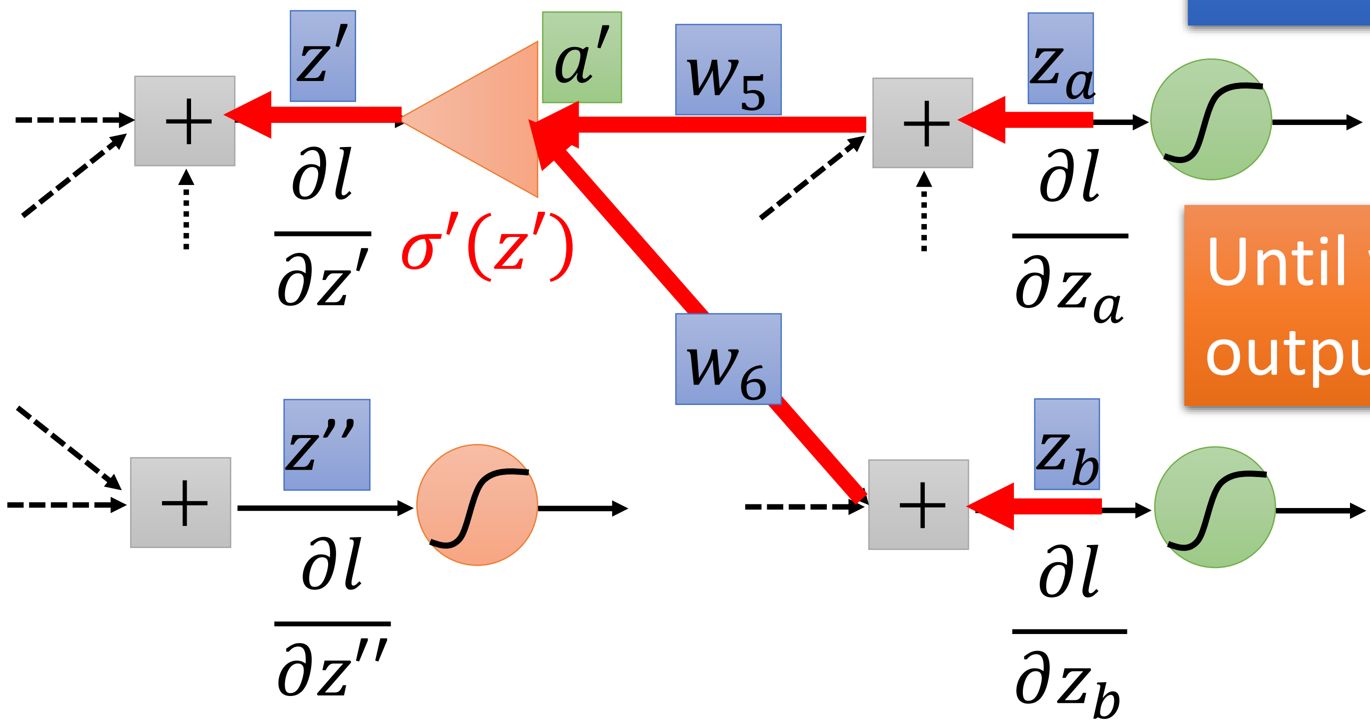
Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z

Case 2. Not Output Layer

Compute $\partial l / \partial z$ recursively

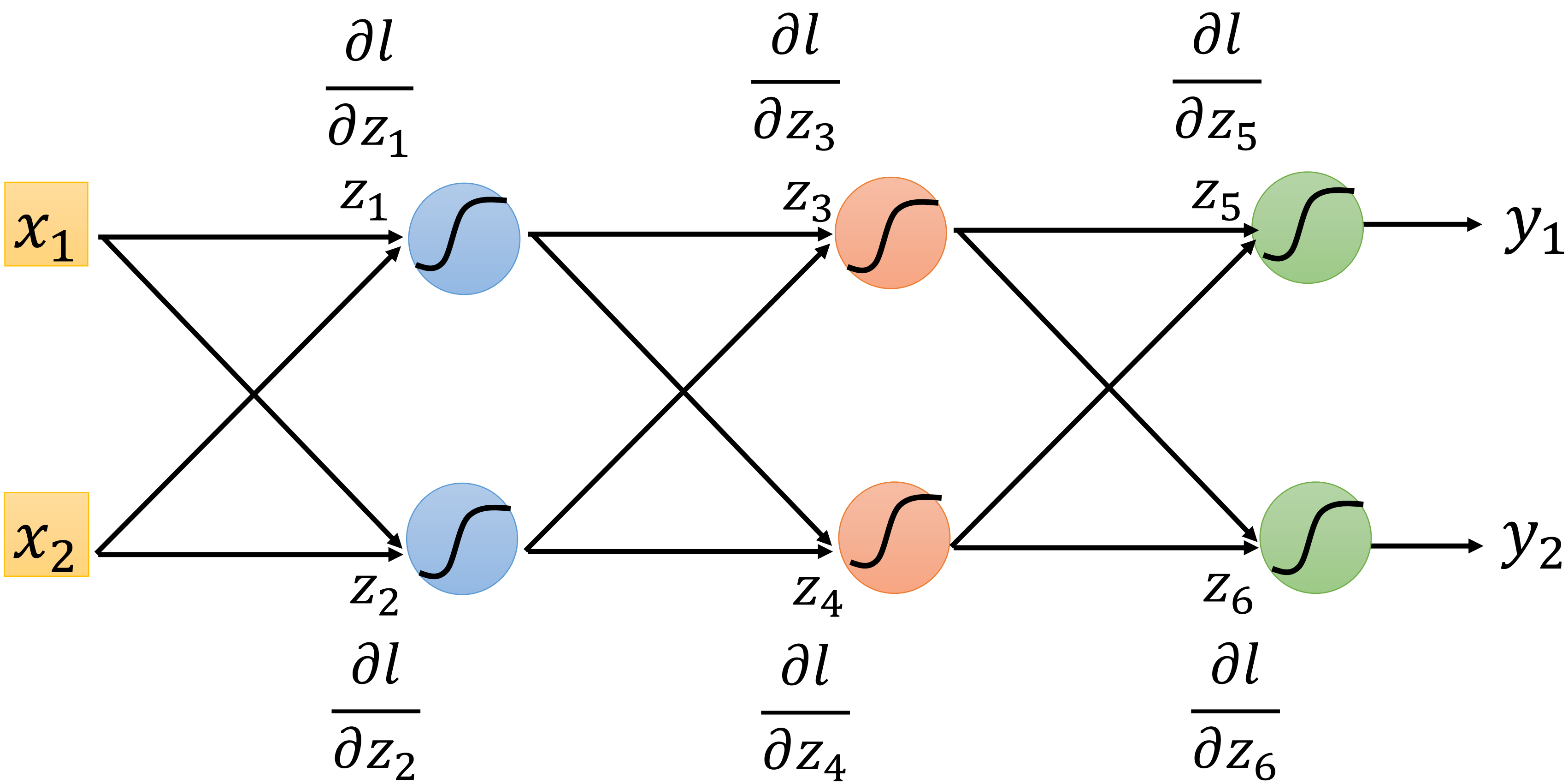
Until we reach the output layer ……



Backpropagation – Backward Pass

Compute $\partial l / \partial z$ for all activation function inputs z

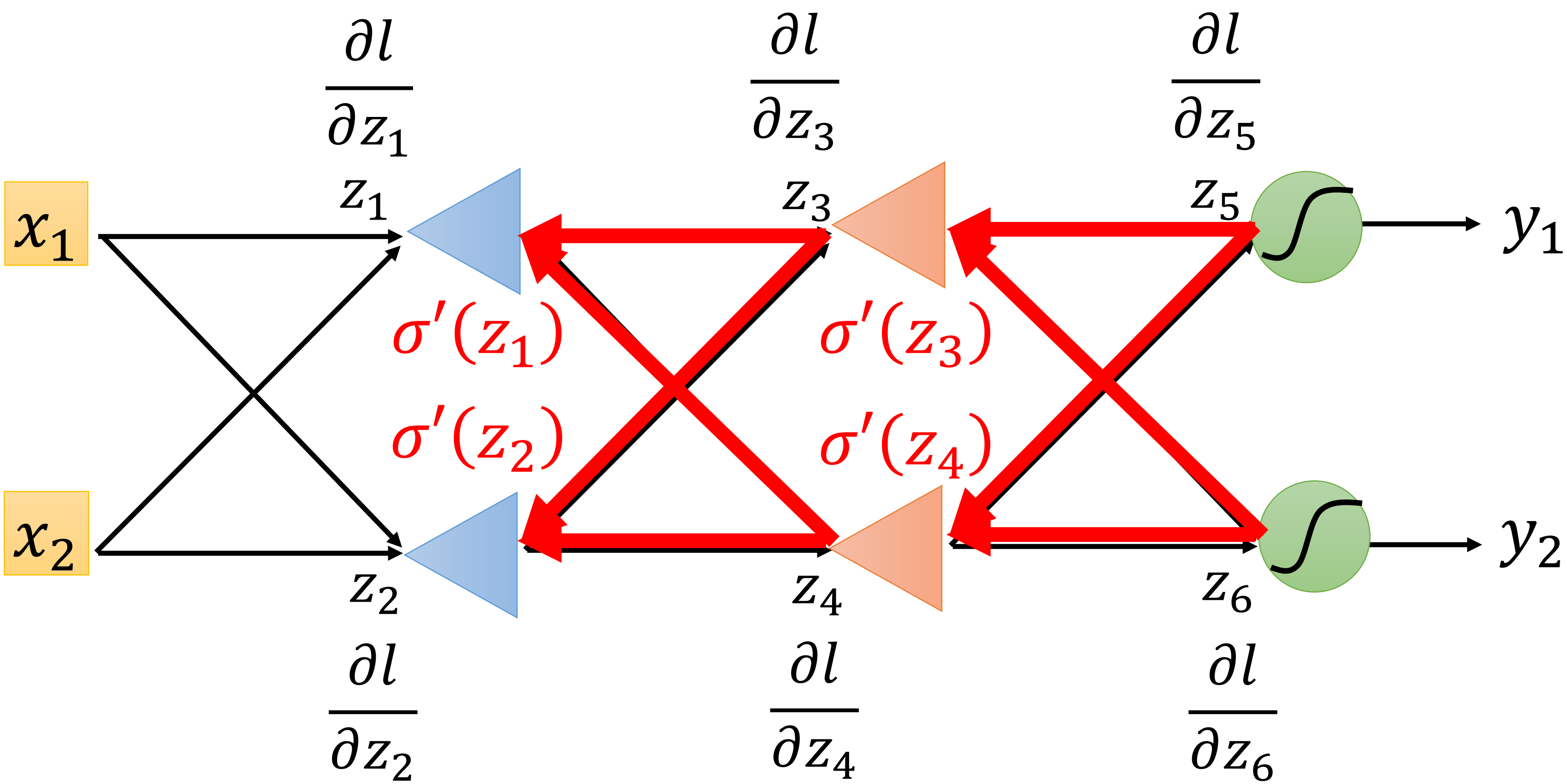
Compute $\partial l / \partial z$ from the output layer



Backpropagation – Backward Pass

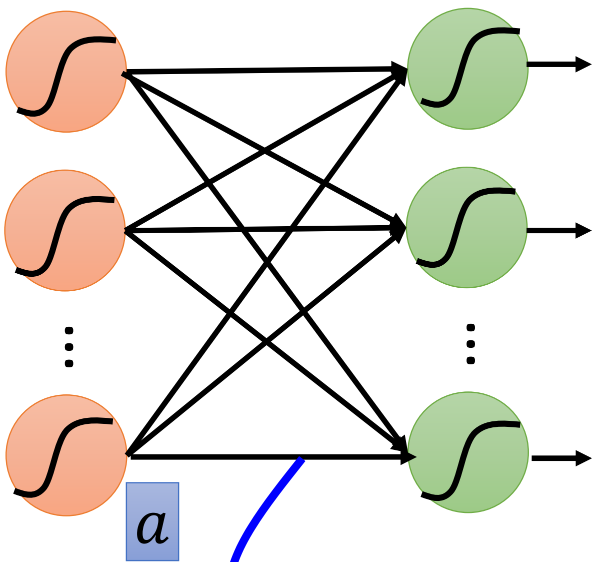
Compute $\partial l / \partial z$ for all activation function inputs z

Compute $\partial l / \partial z$ from the output layer

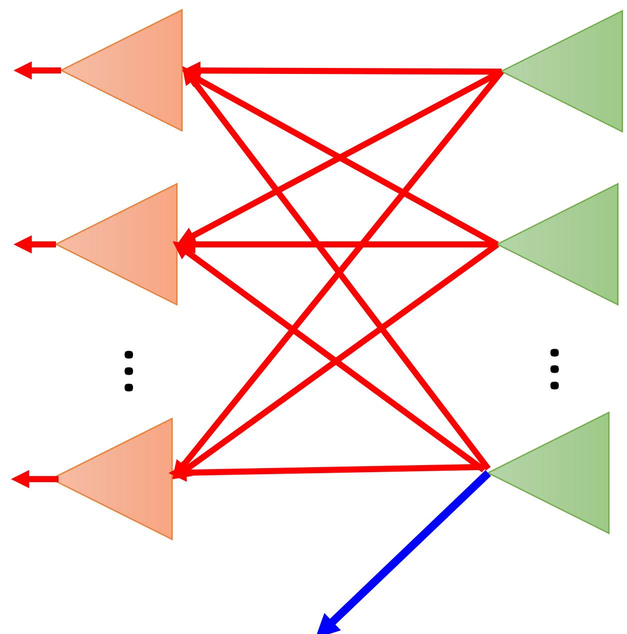


Backpropagation – Summary

Forward Pass



Backward Pass



$$\frac{\partial z}{\partial w} = a \quad \text{X} \quad \frac{\partial l}{\partial z} \quad = \frac{\partial l}{\partial w}$$

for all w